BLOCKING PROBABILITIES IN SMALL COMMUNICATION NETWORKS

by

John McGrath

UNITED STATES
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THESIS

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John Mc Grath
Captain, United States Army
B.S., Northeastern University, 1963

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ABSTRACT

Analytic models of communication networks, combining the desired degree of simplicity and accuracy, are noticeable by their absence from the literature. Modern networks are so large that the combinatorial possibilities impose an overwhelming burden on mathematical model formulation. High speed digital simulations provide one of the only means of solution for the systems planner.

Military communication networks offer more hope for analytical solution. Much smaller than their civilian counterparts, most military systems can be modelled satisfactorily as aggregations of queues. This paper provides approximations to the probability of blocking in 2 terminal, k-channel systems. The probability of blocking is defined as the probability that a vacant channel is available for connecting a caller to his desired party whenever the call is placed.

TABLE OF CONTENTS

Page

CHAPTER I

FORMULATION OF THE PROBLEM

1. Introduction.

Communication systems represent an integral part of progress in the social, business and technical fields. Certainly this statement needs no qualification when applied to the military. With the increase in destructive power and improvement of accuracy in modern weapons, large-scale dispersion of troops on the battle field has become part of the doctrine of warfare. Wider separation has imposed difficulty in controlling and directing forces at all levels of command from squad leader to army commander. Rapid, secure and reliable communication is indispensable in marshalling resources to obtain the military and political objective. Indeed, quality and quantity of communications systems often provides the margin of victory.

Military communication systems encompass a wide range of sophistication; the spectrum runs from the hand-held, short range, single channel radio to the most modern orbiting satellite system. By far the most important in terms of volume of traffic and ease of control is the multi-channel, wide-band radio system in the Army division. This system closely parallels the long-haul, inter city trunking system operated nationally by the Bell System and independent telephone companies. Both of these systems have a network of radio links interconnecting nodal or terminal points.

In the Army division system, subscribers are tied by 2-wire cables to the switching center at the terminal. Local calls are switched and connected to the called party at the local terminal. Calls to other terminals are transmitted over radio links to the desired terminal and are then connected to the 2-wire cable of the desired party.

Civilian networks are operated with automatic switching equipment to route calls while military systems are at present using manual methods in tactical operations. Another difference between the two systems is in the distribution of calls. The civilian system has the bulk of calls contained within the local area whereas the military network has the vast majority of calls going outside the local terminal. This is because subscribers in the military system are located in close proximity to the other subscribers in the same terminal.

Much has been done on various aspects of communication networks. Perhaps the foremost source of analysis associated with developments in this field is the Bell System Technical Journals. In these analyses efforts have been made to arrive at mathematical models admitting an accurate display of the significant features of communications systems. Practical results have been disappointing. The sheer size of a civilian system is staggering. Millions of subscribers tied into terminals that are connected by radio links of up to a thousand channels create immense combinatorial problems. Consequently, most work of a practical nature in the analysis of the operation of networks is done by computer simulation. Ease of solution provides ample compensation for the inelegance and inefficiency of the method. Military systems are more amenable to analytic solution because, as will be seen in Section 1-3, radio links usually have a maximum of 24 channels and rarely does a terminal interconnect with more than two others. Accordingly, this paper will attempt to analytically model the physical operation of a military communications system.

A useful parameter of a communications net is the probability of blocking, P_b. This is defined as the probability that a call cannot be routed so that the desired party is reached, because all available channels of communication are in use. The theory of Markov processes and queues was used to obtain the value of this parameter.

Approximate solutions for P_b will be obtained for a 2 terminal, k- channel system. In addition, an intuitive approach to the solution of a 3 terminal system, again interconnected with k-channel links, will be given. The solution for the 2 terminal model will be preceded by a thorough investigation of a basic system.

2. Organization.

As mentioned above, this paper will attempt to develop an analytic model for finding blocking probabilities for small communications systems. An orderly progression has been attempted by initially treating a very basic model and then, by adding real world complexities, arriving at a useful model for a 2 terminal system. Chapter I gives the background and a broad overview of the communications network within an Army division. The basic model, together with some variations are treated in Chapter II. More complex models, arrived at by considering the work of operators, are introduced in Chapter III. Approximate solutions for the 2 terminal system, interconnected with a k-channel link, are given in Chapter IV. Chapter V provides an intuitive insight into 3 terminal network problems. Areas for further study and extension of this paper are discussed in Chapter VI.

3. The Army Division Communications System.

There is no rigid structure for the division communication system. Many factors must be weighed and evaluated before a concrete plan for a communications system can be drawn up. The type of division (armor, airborne, mechanized, infantry), the mission of the division, the geographical location, the presence or absence of other forces, the personnel strength; all can influence its configuration and design. These factors notwithstanding, there are some general remarks which can be made about the communications support given to the division in the field. The division can have from nine to fifteen battalions. Under present doctrine, the mixture of armor, mechanized infantry or infantry battalions is tailored to the mission and location of the division. Control is exercised by the Commanding General through three brigades whose strength (three to five battalions) also is regulated by local requirements. One brigade is normally held in reserve.

The signal battalion, which is organic to the division, must furnish communication support for the division to Corps and Army HQS and to the main sub-ordinate headquarters. Also, within the division area of responsibility, signal centers must be set up to allow small units to connect into a division-wide communications network. Many modes of communication are employed. Hand-held radios suffice at the squad level; however as the size of the unit and the physical separation increases, more sophisticated modes are required. These include line of sight microwave and tropospheric scatter. While many advances have been made in the past ten years, it is still true that within the division the main communications system is a twelve or twenty-four channel microwave radio system manually controlled by operators at switchboards. Figure 1 shows how a typical division system might be configured. Usually every terminal has access to Division Main and one other terminal. This is a precaution against loss of a link due to

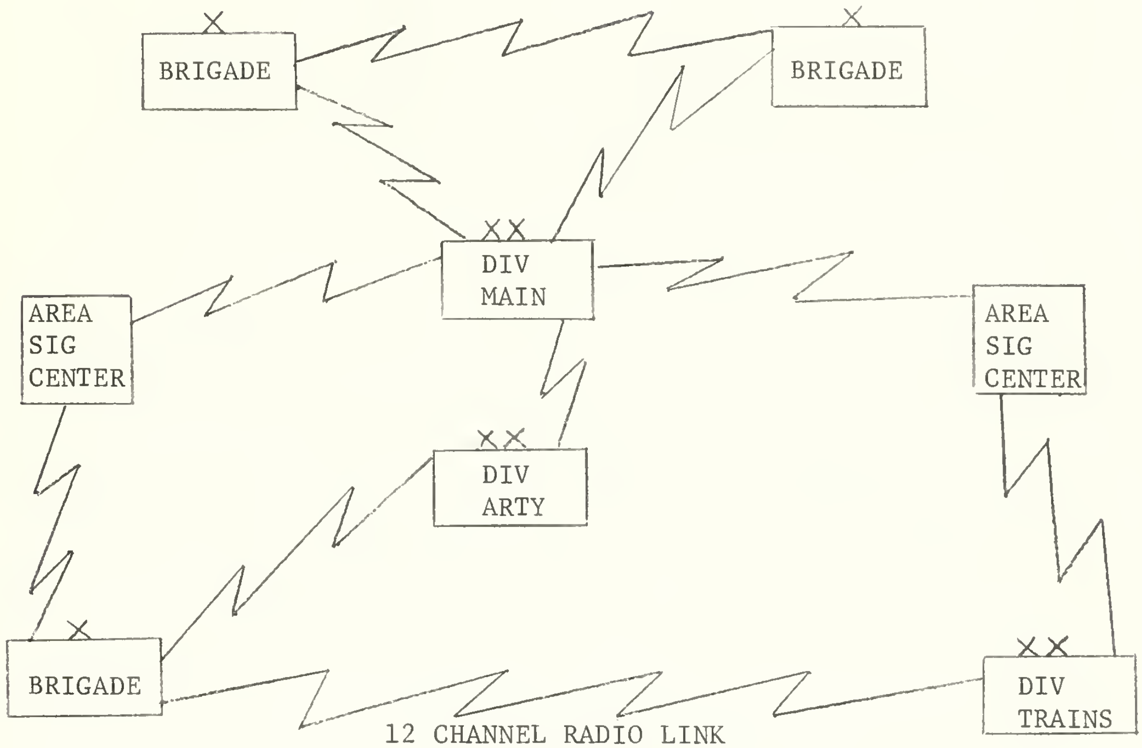


FIGURE 1

equipment failure or enemy action. Division Main, normally the location of the Commanding General and his staff is connected with as many other terminals (at least the brigades and division artillery) as equipment and personnel allow.

At each terminal an operator is on duty to receive requests for service, coordinate with other operators in routing the call, and disconnect the line when the call is complete. As shown in Figure 2, each

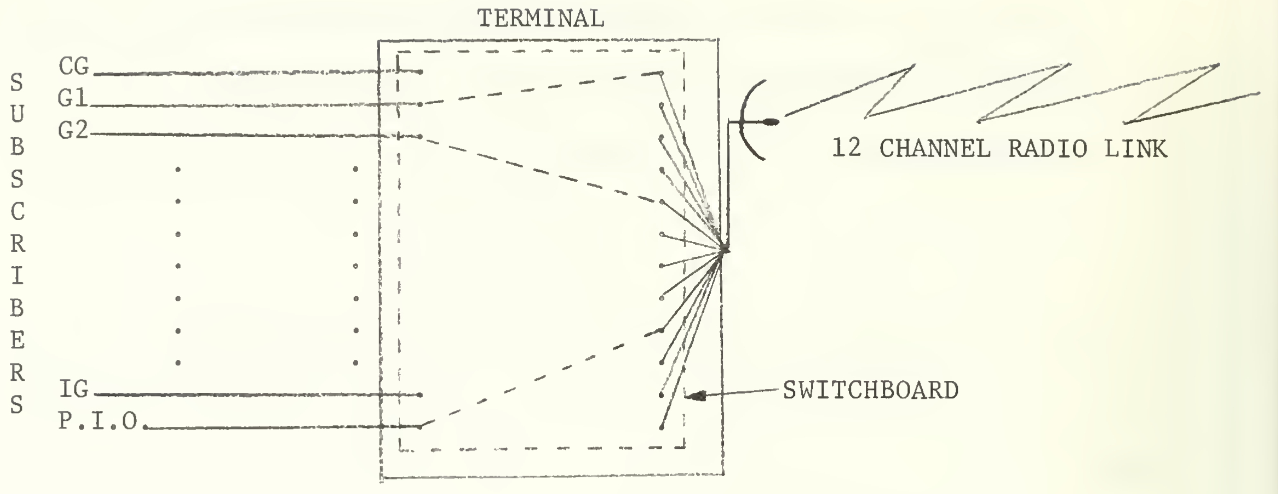


FIGURE 2

subscriber has an individual connection to the switchboard on the "subscriber" side of the board. At the switchboard, a light appears above the termination of a subscribers line when he turns a crank on his telephone set. When the operator responds to the request for service he determines with whom the subscriber wishes to talk. He then rings the operator at the terminal serving the requested party. This requires that at least one channel in the radio link between operators is not being used. When the distant operator has reached the requested party, the first operator tells his subscriber that the connection has been completed and he can commence his call.

CHAPTER II

ANALYSIS OF A SIMPLE NETWORK DISREGARDING OPERATOR PATCH TIME

1. Basic Model.

Before analyzing more complex models an understanding of the operation of a basic system is helpful. Accordingly the most trivial network will be examined first. A simple communications system is shown in Figure 3. Here we have only one channel available for use

FIGURE 3

between terminal A and B. The following assumptions will be made:

a) Requests for service from the subscriber arrive at each switchboard with a Poisson distribution, mean rate α.

b) The time taken to complete a conversation from the end of routing to hang-up, the "holding" time for the call, is distributed exponentially with mean σ.

c) Time between operator answering subscriber and the call commencing (the operator "patching" time) is assumed negligible in comparison to $\frac{1}{\sigma}$.

d) Having signalled the operator, subscribers wait until the operator answers. In other words, once a subscriber decides to make a call, he is not discouraged by the time required to have the operator answer.

The first two assumptions are reasonable and have been verified by a study on arrival rates and holding times in a civilian telephone exchange in New Jersey. [1]

This system can be treated as a basic queueing model with exponential inter-arrival time and one server. The literature generally denotes this as an M/M/1 queue. A steady-state solution for blocking probability can be obtained if the utilization rate, ρ , is less than unity where

$$\rho = \frac{2\alpha}{\sigma} \tag{1}$$

Steady-state implies that the system is in operation for a long period of time and consequently the probability of being in a particular state of the stochastic process is independent of the state of the process when first observed. This guarantees that expected system size (i.e., expected number waiting for service plus those being serviced) is always finite. The probability of blocking, p_b , in this system is the probability of the channel being busy and is equal to

$$p_b = \rho = \frac{2\alpha}{\sigma} \tag{2}$$

The probability p_i of $(i - 1)$ callers waiting for service is

$$p_i = (1 - \rho)\, \rho^i \tag{3}$$

It is worth noting that in this system, queue discipline is not necessarily first come, first served (FCFS). Normally when queue discipline is not specified it is FCFS. Here, service is not necessarily rendered according to order of arrival. For example, the first arrival at switchboard A, with the channel busy, may occur before the first arrival at switchboard B. But if the operator at B becomes aware of the termination of a call before his opposite number at A then the first subscriber in line at B will be served next although some have been waiting longer at A. This however is an internal organization problem and, while it is of interest to the individual subscriber, it does not alter the blocking probability.

2. Two terminals connected by a k-channel link.

Consider now a system consisting of two terminals with k channels available for communication between them. Figure 4 illustrates such a system. The assumptions made in the previous section will be

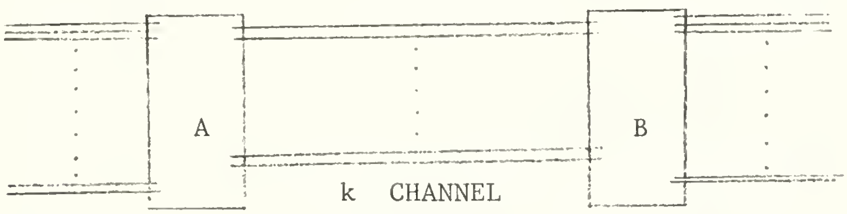


FIGURE 4

retained. Implicit in these assumptions is the understanding that the number of lines on the subscriber side of the board is so large that a) the future arrival rate, α , of calls is not affected by the number of calls presently initiated and b) the probability of a called party being busy can be assumed to be zero.

The system, therefore, can be modeled as an M/M/k queue. That is, inter-arrival time exponentially distributed with mean $\frac{1}{\alpha}$, service time exponentially distributed with mean $\frac{1}{\sigma}$, and k servers. Define p_n as the probability that system size is n. Again note that system size includes both subscribers waiting for service and being served. The probability p_n that system size is n, then can be expressed as

$$p_n = \begin{cases} \dfrac{p_o \left(\dfrac{2\alpha}{\sigma}\right)^n}{n!} & n \leq k \\ \dfrac{p_o \left(\dfrac{2\alpha}{\sigma}\right)^n}{k!\ k^{n-k}} & n > k \end{cases} \tag{4}$$

where $p_o = 1 - \sum_{n=1}^{\infty} p_n$

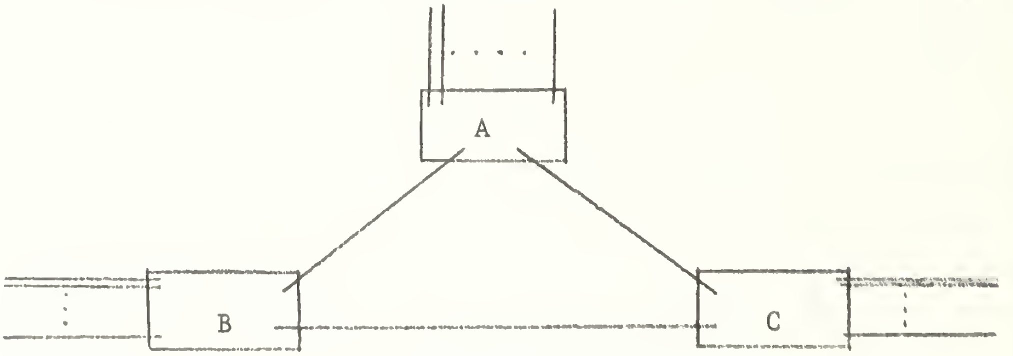
The probability of blocking, p_b, is

$$p_b = 1 - \sum_{n=0}^{k-1} p_n \tag{5}$$

for this system.

3. Three terminals interconnected by 1 channel links.

A representation of this system is shown in Figure 5. Again the



1 CHANNEL
RADIO LINK

FIGURE 5

assumptions previously noted will be retained. One additional requirement for this system is that a call can be routed over one link only. Here we denote the mean arrival rate at terminal i of calls going to terminal j as α_{ij} . With operator patch time assumed zero and maximum route length of one link, the system may be modeled by three independent M/M/1 queues with

$$\rho_1 = \frac{\alpha_{AB} + \alpha_{BA}}{\sigma} \tag{6}$$

$$\rho_2 = \frac{\alpha_{AC} + \alpha_{CA}}{\sigma} \tag{7}$$

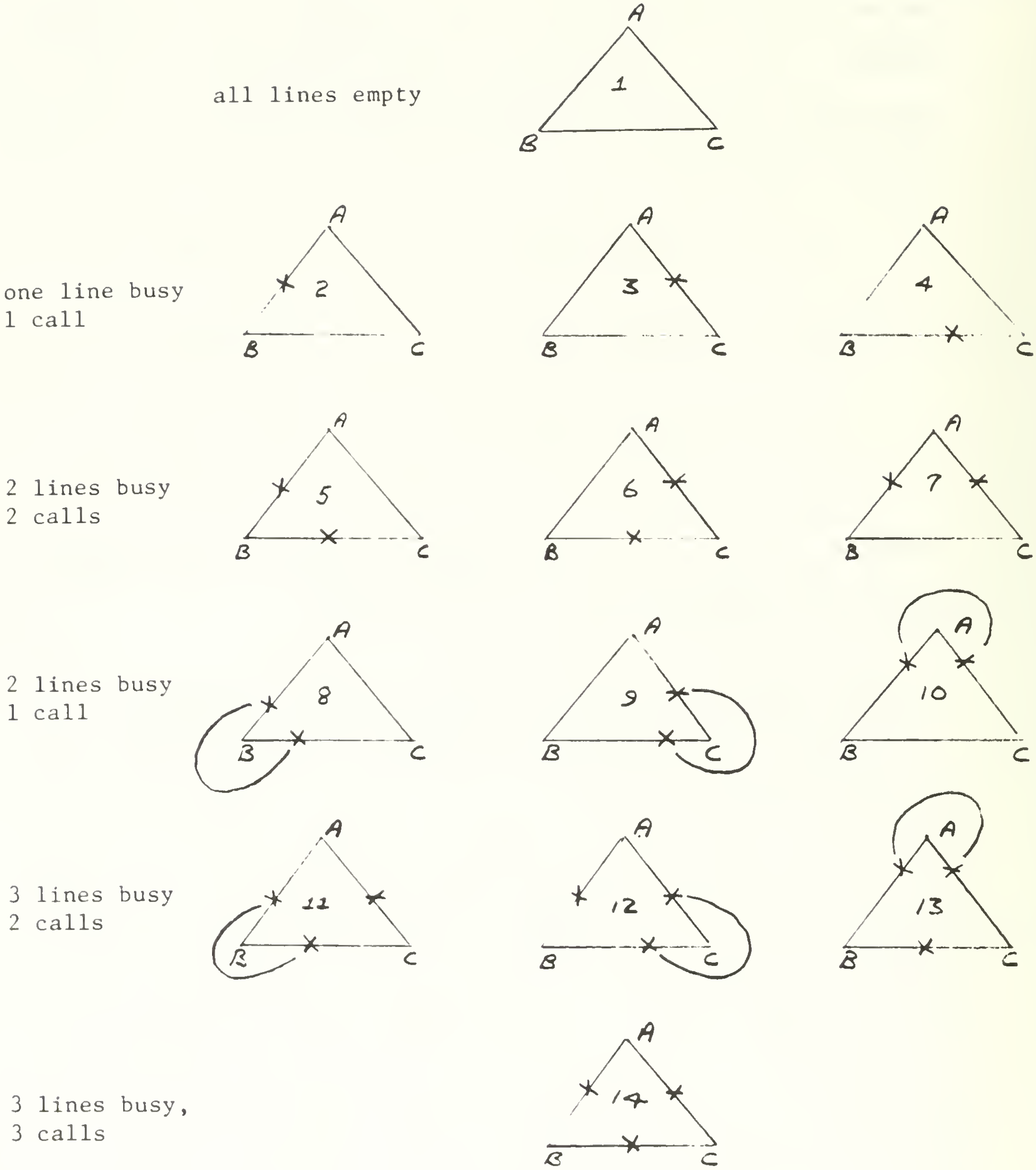
$$\rho_3 = \frac{\alpha_{BC} + \alpha_{CB}}{\sigma} \tag{8}$$

As before the order of arrival at the switchboard for any of these queues is not necessarily the order of service.

4. Three terminal system with alternate routing available.

Introduction of alternate routing gives a degree of flexibility to the previous model as well as adding realism. Alternate routing allows a call from A to B to be routed on links A-C and C-B, if they are free and link A-B is busy. Here assumptions made in Section 1 above hold with the exception that if all links are busy incoming callers hang up. For ease in computation assume all arrival rates, α_{AB}, α_{BA}, . . . , α_{CB} are the same and equal to $\alpha/2$. Figure 6 depicts all possible states of the system. A state is a unique position which completely describes the process at any instant of time.

It is now possible to obtain a transition rate matrix A, which gives the rate of movement between states of the system. Solving the



POSSIBLE STATES OF THE SYSTEM

FIGURE 6

system of equations

$$\vec{P}\,A = \vec{0} \tag{9}$$

where $\vec{P} = (p_1\ p_2\ .\ .\ .\ p_{14})$ p_i, the steady state probability that the system is in state i can be obtained. Figure 7 shows the transition matrix for this system. Note that it is possible to reduce the transition matrix from 14 x 14 to 6 x 6 by grouping similar states together in one "super-state".

For example: states 2, 3, and 4 may be combined to create the super-state — "one call, one line busy". Using equation (8), a set of simultaneous equations can be written as follows:

$$\begin{aligned}
-\alpha\, p_1 + \sigma\, p_{II} + \sigma\, p_{IV} &= 0\\
\alpha\, p_1 - (\alpha+\sigma)\, p_{II} + \frac{2}{3}\sigma\, p_{III} + \frac{\sigma}{3}\, p_V &= 0\\
\frac{2}{3}\alpha\, p_{II} - (\alpha + \frac{2}{3}\sigma) p_{III} + \sigma\, p_{VI} &= 0\\
-(\sigma + \frac{\alpha}{3})\, p_{IV} + \frac{\sigma}{3}\, p_V &= 0\\
\frac{\alpha}{3}\, p_{II} + \frac{\alpha}{3}\, p_{IV} - \frac{2}{3}\sigma p_V &= 0\\
\alpha\, p_{III} - \alpha\, p_{VI} &= 0\\
p_I + p_{II} + p_{III} + p_{IV} + p_V + p_{VI} &= 1
\end{aligned} \tag{10}$$

FIGURE 7

TRANSITION MATRIX

		1	2	3	4	5	6	7	8	9	10	11	12	13	14
I	1	-3α	α	α	α										
II	2	σ	$(-3\alpha-\sigma)$			α		α					α		
	3	σ		$(-3\alpha-\sigma)$			α	α				α			
	4	σ			$(-3\alpha-\sigma)$	α	α							α	
III	5		σ		σ	$(-\alpha-2\sigma)$									α
	6			σ	σ		$(-\alpha-2\sigma)$								α
	7		σ	σ				$(-\alpha-2\sigma)$							α
IV	8	σ							$(-\alpha-\sigma)$			α			
	9	σ								$(-\alpha-\sigma)$			α		
	10	σ									$(-\alpha-\sigma)$			α	
V	11			σ					σ			(-2σ)			
	12		σ							σ			(-2σ)		
	13				σ						σ			(-2σ)	
VI	14					σ	σ	σ							(-3σ)

	I	II	III	IV	V	VI
I	$-\alpha$	α				
II	σ	$-\alpha-\sigma$	$\frac{2\alpha}{3}$		$\frac{\alpha}{3}$	
III		$\frac{2}{3}\sigma$	$(-\alpha-2\sigma/3)$			α
IV	σ			$(-\alpha/3-\sigma)$	$\frac{\alpha}{3}$	
V		$\frac{\sigma}{3}$		$\frac{\sigma}{3}$	$-\frac{2\sigma}{3}$	
VI			σ			$-\sigma$

As an example, let $\frac{\alpha}{\sigma} = \frac{1}{2}$. This yields the following results

$$
\begin{aligned}
p_{I} &= \text{probability of no lines busy} = \frac{112}{221} \\
p_{II} &= \text{probability of 1 line busy} = \frac{52}{221} \\
p_{III} + p_{IV} &= \text{probability of 2 lines busy} = \frac{30}{221} \\
p_{V} + p_{VI} &= \text{probability of 3 lines busy} = \frac{27}{221}
\end{aligned}
\tag{11}
$$

For this system the probability of blocking, p_b, with $\frac{\alpha}{\sigma} = \frac{1}{2}$ is

$$
\begin{aligned}
p_b &= \text{prob } \{3 \text{ lines busy}\} + \frac{2}{3} \cdot \text{prob } \{2 \text{ lines busy}\} \\
&= \frac{27}{221} + \frac{20}{221} = \frac{57}{221} = .259
\end{aligned}
\tag{12}
$$

For the same system with 2 channels per link connecting every pair of terminals, there are 34 possible states. Using symmetry and grouping similar states as before, a solution can be obtained from a set of 11 simultaneous equations. It can be recognized that even this simple problem requires a significant amount of time to solve manually if computing equipment is not available.

5. Extensions to multi-terminal networks

It would be useful if generalizations of the results of the previous section were available for n-terminal networks interconnected with k-channel links. No such solution seems to be available. It appears that systems of interconnected terminals must be treated individually and, except for the most trivial, the combinatorial aspects make analytic solutions appear unpalatable. Benes [3] states "the performance of a connecting network for a given level of offered traffic is determined largely by its configuration or structure ...". This statement certainly has intuitive appeal and an example can exhibit its validity. Consider two networks, with the same number of terminals and total channels available for use, as shown in Figure 8. They differ only in network configuration. Nevertheless, certain states

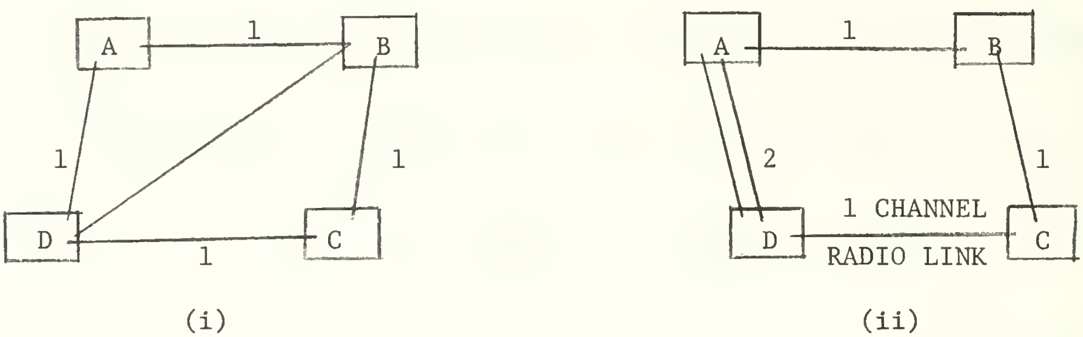


FIGURE 8

will have different probabilities. For example consider the state "2 lines busy, 1 call ". For (i) there are eight ways to obtain this condition while in (ii) there are only six ways to accomplish the same condition even though the number of terminals and total available channels are the same. Obviously with different flow rates the systems would be incompatible.

CHAPTER III

MODELS FOR NETWORKS WITH NON-ZERO PATCH TIME

1. Two terminal system with operator patch time.

A more complex system to analyze is one similar to that shown in Figure 3 but in which operator patch time is considered significant. Again, we assume mean arrival rates of calls at both switchboards of α and holding time for calls with mean rate σ. In addition operator patch time is taken as distributed exponentially with parameter μ. Queue discipline is nominally first come, first serve and callers wait for service if the channel is occupied.

Having only a one channel link between terminals makes available for this system a special analytical solution. Service time is the sum of two independent random variables distributed exponentially but with different means. Normally once an operator has completed a patch he is free to assist the next caller in line at the switchboard. In this system, however, as there is only one channel the operator is idle until the call occupying the channel is terminated. In practice the operators arrange the routing of the requested call on the same channel that the call itself will use.

The system can be modeled as an M/G/1 queue with arrival rate distributed exponentially with mean 2α. Service time, $B(t)$, has a mean and variance

$$m = \frac{1}{\sigma} + \frac{1}{\mu} = \frac{\sigma + \mu}{\sigma\mu} \tag{1}$$

$$v = \frac{1}{\sigma^2} + \frac{1}{\mu^2} = \frac{\sigma^2 + \mu^2}{\sigma^2\mu^2} \tag{2}$$

The embedded Markov chain technique of Kendall[4] can be used to solve this type of queue and obtain answers to the most interesting questions (e.g. number of callers awaiting service, probability of having x callers awaiting service, expected waiting time for service). Define

X_n , n 0	the number of calls awaiting service when n^{th} call is completed
U_n , $n \geqq 0$	the number of calls arriving at the switchboards during the routing and calling of the $n + 1^{st}$ call.
u_k , $k \geqq 0$	prob $\{U_n = k\}$
ρ	traffic intensity
$G_y^*(z)$	z transform of y
$F_y^*(s)$	Laplace transform of y
y_i	steady state probability that $x_n = i$

We know[5] that $\{x_n\}$ is a recurrent Markov chain if

$$G_{u_k}^{*'}(1) = \sum_{k=0}^{\infty} k\, u_k \leqq 1 \tag{3}$$

but,

$$G_{u_k}^{*'}(1) = E[U_n] = \alpha\, m = \rho \tag{4}$$

and so if ρ is less than unity, steady-state solutions exist. It can be shown that

$$G_{y_i}^{*}(z) = \frac{(1 - \rho)\ (1 - z)\ F_{B(t)}^{*}\ [2\alpha\ (1 - z)]}{F_{B(t)}^{*}\ [2\alpha\ (1 - z)]\ - z} \tag{5}$$

where
$$F_{B(t)}^{*}\ [2\alpha\ (1 - z)] = \frac{\sigma \qquad \mu}{[\sigma + 2\alpha\ (1 - z)]\ [\mu + 2\alpha\ (1 - z)]} \tag{6}$$

in this example, and

$$y_o = G^{*}_{y_i}(z)/z=0 = 1 - \rho$$

$$y_1 = G^{*'}_{y_i}(z)/z=0 \qquad (7)$$

$$y_i = G^{*i}_{y_i}(z)/z=0 \text{ , etc.}$$

To illustrate the solution technique the following example is given:

$$\text{let } \alpha = \frac{5}{2} \text{ per hour}$$

$$\mu = 50 \text{ per hour}$$

$$\sigma = 10 \text{ per hour}$$

$$\text{then } \rho = 5 \cdot \frac{50 + 10}{500} = \frac{6}{10} < 1$$

and steady state conditions exist.

$$y_o = 1 - \rho = .4$$

$$p_b = 1 - y_o = .6$$

$$\text{Prob}\{\text{more than 1 call at the switchboard}\} = 1 - y_o - y_1$$

$$= 1 - .4 - .23 = .37$$

Expected system size is obtained from

$$E[x_n] = \frac{\rho + \rho^2 + \alpha^2 v}{2(1 - \rho)}$$

$$= \frac{11}{8} = 1.375$$

$$\text{Operator idle time per hour} = y_o + \frac{1}{2} y_1$$

$$= .515 \text{ hours}$$

Expected total time in system $= \frac{1}{2\alpha} E(x_n)$

$= .275$ hours

Expected time waiting for an operator $= .275 - m$

$= .156$ hours

2. Two terminal system with one way traffic flow.

A much more difficult system to analyze is that shown in Figure 4 with the following additional constraints. All traffic is assumed to be initiated at one terminal and exponential operator patch time (with mean rate u) is considered. Now once the operator has completed a patch from A to B he can immediately begin to service the next customer (if there is one) at the switchboard providing at least one of the k lines is vacant.

This system can be described as a two stage queue. Stage 1 is the operator patching operation and is an M/M/1 queue with infinite waiting room. Stage 2 is the holding time per call and is a G/M/n, $n \leqq k$, queue with no waiting permitted.

The literature includes work done in the area of sequential array or tandem queues.[6],[7] This system, however, is unlike those treated in the literature in that the stages are not alike. They have a strong dependence on each other, unlike the usual tandem queues which involve movement from one stage to the next and where the only dependence of "successor" stages is upon their predecessor's output. This system has the feature that while Stage 2 takes its arrival rate as the service output of Stage 1, the service rate in Stage 1 also depends upon the system size of Stage 2.

A solution technique for this system is as follows:
Let r denote the number of channels busy (in either patching or conversation). Then for Stage 1, service rate for a customer is

$$\bar{u} = \begin{cases} u & r < k \\ 0 & r = k \end{cases} \tag{8}$$

Hunt[7] shows that for a 2 stage system, both stages being M/M/1 with unlimited waiting room, the probability of system size being n is

$$p_n(z) = \sum_{i=0}^{n} p_i^{(1)} p_{n-i}^{(2)}$$

$$= (n + 1)(1 - \rho)^2 \rho^n$$

where $p_i^{(j)}$ = probability {i calls in stage j at steady state}

$p_i(R)$ = probability {i calls in an R-stage system}

Stage 1 is recurrent (i.e., $\{p_i^{(1)}\}$, $i \geq 0$ is a recurrent Markov chain) if ρ, the traffic intensity or utilization rate, is strictly less than unity. Normally in an M/M/1 queue $\rho = \frac{\alpha}{u}$, however, in this case service rate depends upon whether any lines are empty. We can therefore approximate u, the expected mean service rate of Stage 1 to be

$$\bar{u} = u \cdot \text{prob}\{r < k\} + 0 \cdot \text{prob}\{r=k\} \tag{9}$$

$$= u \cdot \text{prob}\{r < k\}$$

a constant.

Stage 1 is recurrent if $\alpha < \bar{u}$. Then the following statement is true: If Stage 1 is a recurrent Markov chain the rate of departure from Stage 1 must equal the arrival rate α.[8] The distribution of the inter-departure times is no longer exponential due to the change in service time distribution. However, for many problems of interest the output of Stage 1 can be approximated as Poisson without incurring great error. This is because in any real-life system the prob $\{\#$ of lines busy $= k\}$ is small.

A series of simulations were run on an IBM 360/67 computer to verify this Poisson approximation. An internal package called GPSS[9] was used to generate 3 runs of a 2 stage system similar to that described above. 1000 inputs were used in each run and reasonable values of α, μ and σ were selected. Data from these simulations are shown in Table 1. It can be seen that the inter departure times of the output of Stage 1, α_2, have the general characteristics of an exponential distribution (i.e., mean and standard deviation are the same). In addition the mean rates of input and output of Stage 1 are approximately the same. By way of contrast, 2 runs were generated with parameter values selected so that the system was unstable. Runs #4 and #5 of Table 1 show the difference in input and output ratio in Stage 1 when the service rate of Stage 2 is equal to input rate of Stage 1 (run #4) and when the service rate of Stage 2 is less than the arrival rate to Stage 1 (run #5).

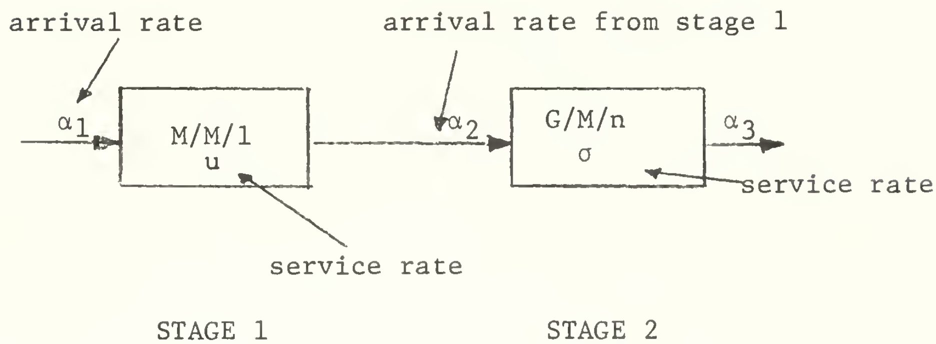
In Stage 2 the probability of having 2 in the system is

$$p_i^{(2)} = \frac{1}{\sum\limits_{j=0}^{k} \frac{\rho_2^{\,j}}{j!}} \; \frac{\rho_2^{\,i}}{i!} \tag{10}$$

$$= \frac{\rho_2^{\,i}}{i! \sum\limits_{j=0} \frac{\rho_2^{\,j}}{j!}}$$

PARAMETERS

RUN	$\frac{1}{\alpha_1}$	$\frac{1}{u}$	$\frac{1}{\sigma}$	n	$\frac{1}{\alpha_2}$ mean	$\frac{1}{\alpha_2}$ stand. dev.	
1	10	2	10	6	9.74	9.79	
2	3	2	10	6	2.61	2.69	
3	5	2	10	6	4.45	4.56	
4	5	1	30	6	5.94	5.88	
5	3	1	40	6	6.84	7.02	



NOTE: numbers in table represent time units between arrivals, departures or services

RESULTS OF SIMULATION OF 2-STAGE QUEUE

TABLE 1

and $\quad \text{Prob}\ \{r=k\} = \dfrac{\rho_2^{\,k}}{k! \displaystyle\sum_{j=0}^{k} \dfrac{\rho_2^{\,j}}{j!}}$ (11)

where $\quad \rho_2 = \dfrac{\alpha}{\sigma}$

In Stage 1 the system utilization rate is

$$\rho_1 = \frac{\alpha}{u - \dfrac{u\,\rho_2^{\,k}}{k! \displaystyle\sum_{j=0}^{k} \dfrac{\rho_2^{\,j}}{j!}}}$$

$$= \frac{\alpha\, k! \displaystyle\sum_{j=0}^{k} \dfrac{\rho_2^{\,j}}{j!}}{u\, k! \displaystyle\sum_{j=0}^{k} \dfrac{\rho_2^{\,j}}{j!} - u\rho_2^{\,k}} \tag{12}$$

and $\quad p_i^{(1)} = (1 - \rho_1)\, \rho_1^{\,i}$ (13)

Therefore for the 2 Stage system

$$p_n^{(2)} = \begin{cases} \displaystyle\sum_{i=0}^{n} p_i^{(1)}\, p_{n-i}^{(2)} & n < k \\ \displaystyle\sum_{i=0}^{k} p_{n-k+i}^{(1)}\, p_{k-i}^{(2)} & n \geqq k \end{cases} \tag{14}$$

and by using equations 8 through 14 we get

$$
p_n^{(2)} = \begin{cases} \displaystyle\sum_{i=0}^{n} (1-\rho_1)\,\rho_1{}^{i} \frac{\rho_2{}^{n-i}}{(n-i)!\displaystyle\sum_{j=0}^{k} \frac{\rho_2{}^{j}}{j!}} & ,n < k \\[2ex] \displaystyle\sum_{i=0}^{k} (1-\rho_1)\,\rho_1{}^{(n-k+i)} \frac{\rho_2{}^{k-i}}{(k-i)!\displaystyle\sum_{j=0}^{k} \frac{\rho_2{}^{j}}{j!}} & ,n \geqq k \end{cases} \tag{15}
$$

and

$$
P_b = (1-\rho_1) \sum_{i=0}^{k} \frac{\rho_1{}^{i}\,\rho_2{}^{k-i}}{(k-i)!\displaystyle\sum_{j=0}^{k} \frac{\rho_2{}^{j}}{j!}} \tag{16}
$$

is the probability of blocking.

CHAPTER IV

2 TERMINAL SYSTEMS WITH PRIORITIES AND UNLIMITED ACCESS

1. Models with unlimited access.

A much more complex system to analyze is that in Figure 4 when arrivals from subscribers are allowed at both switchboards. As discussed in Section 2, Chapter III the system can be modeled as a tandem network consisting of a M/M/1 queue and a G/M/k queue with no waiting room. A subtle distinction should now be noted. The model is now susceptible to total collapse under the following circumstances. Having assumed Poisson distributed arrivals from subscribers at both switchboards, the probability of instantaneous arrivals at both switchboards is zero. However, consider what would happen if the time between arrivals is small compared to the operator patch time. Operator B could have begun patching a subscriber before operator A could patch his subscriber to B. Thus the system would be suspended in a state of inaction with both operators trying to contact each other. Some real world networks prevent this situation from occuring by designating one channel for use by operators. Hence both operators would never be trying to get in touch with each other at the same time. Use of the "orderwire", as the designated line is sometimes called, is inefficient because only k-1 channels can be used for communication from A to B. In most systems, if the problem arises, it is disposed of by putting a higher priority on calls arriving at the switchboard from another operator than on calls from the subscriber side of the switchboard. If an operator is busy with one of his subscribers and notices the other operator ringing he immediately stops service and attends to the priority call.

Since the mean rate of calls at both switchboards is assumed to be $\frac{\alpha}{2}$, each switchboard in effect receives $\frac{\alpha}{2}$ high priority and $\frac{\alpha}{2}$ low priority calling rates. Operator patching time can no longer be considered an integral block but must be reduced to its two component phases: the work done by each operator. Without much error, their rates can be considered equal ($u_A = u_B = u$). The system can then be depicted as shown in Figure 9. Note that exit from queues A and B

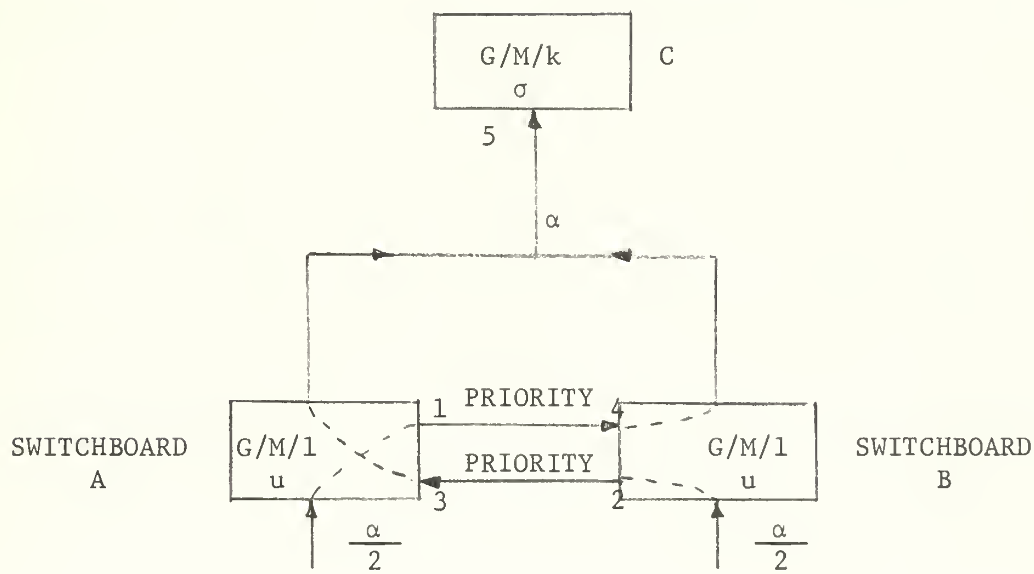


FIGURE 9

at 1 and 2 respectively is allowed only if the number being served at 3 , 4 and 5 plus the number waiting at 3 and 4 is less than k. Queue C has no waiting(as indicated by k servers). Once entry is obtained to queue C service begins immediately. This corresponds to the fact that a call is not routed unless an empty channel is available. Once the routing is complete the call can begin (queue C) without delay.

2. <u>An approximation of the system.</u>

Analysis of this system is extremely complex. It combines in varying degree such features as tandem queues with restricted waiting facilities, networks of queues, and priority queues with general arrival distributions. All of these topics have been discussed individually in the literature and solutions for some of the more useful properties obtained.

The following approximation would be helpful in obtaining blocking probabilities for this system: Assume that the inter departure time from the queues for patching is distributed exponentially. It is obvious that in steady state the output of the first queue (operator) is no longer Poisson. Although arrivals from the subscriber side enter at a Poisson rate, because they have lower priority and because service is sometimes interrupted when k lines are full, they leave at a mean rate of $\frac{\alpha}{2}$ but with unknown, general distribution. If, however, the total arrival rate α is small compared to $\bar{u}$ (effective service rate, see Section 2, Chapter III) the distribution of outputs entering the second switchboard queue should be approximately Poisson.

To show that this approximation can be very reasonable, a queue having both low priority and high priority arrivals was simulated. Both types of arrivals were Poisson with mean rate 15 per hour. Service rate was exponential with mean time of 1 minute. The output distribution is shown in Figure 10. Because of the low utilization rate, 0.5, the inter departure time of the low priority calls has the shape of an

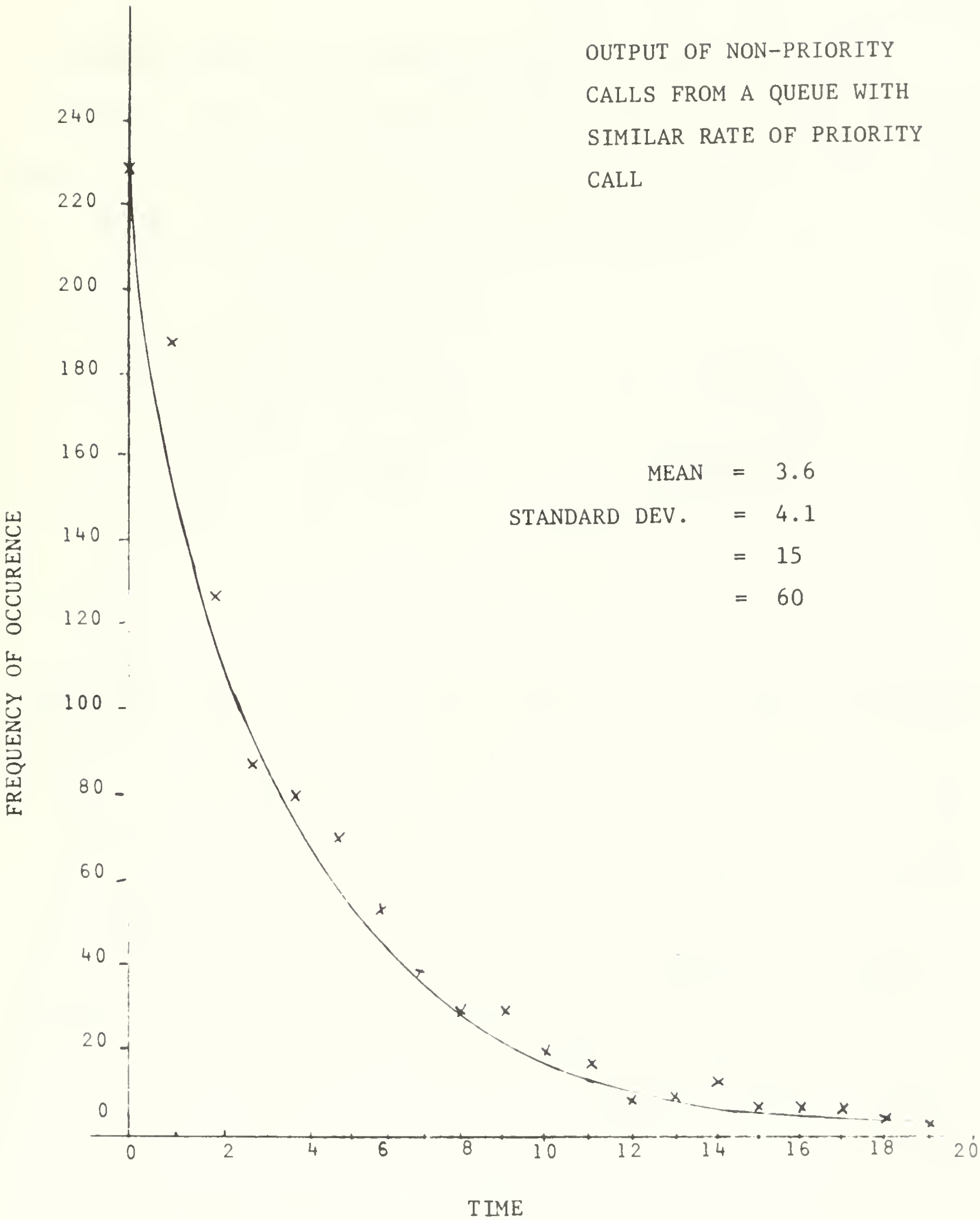

OUTPUT OF NON-PRIORITY
CALLS FROM A QUEUE WITH
SIMILAR RATE OF PRIORITY
CALL
MEAN = 3.6
STANDARD DEV. = 4.1
= 15
= 60
FREQUENCY OF OCCURENCE
240
220
200
180
160
140
120
100
80
60
40
20
0
0
2
4
6
8
10
12
14
16
18
20
TIME

FIGURE 10

exponential.

If this approximation is used the system can be modeled as in Figure 11. It is clear that only one operator can be busy handling calls from the line side of the switchboard at any one time. Assuming that arrivals at the switchboard from the line side are Poisson

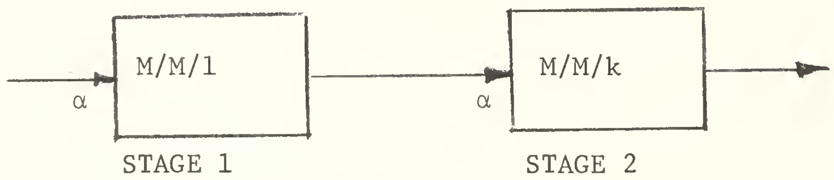
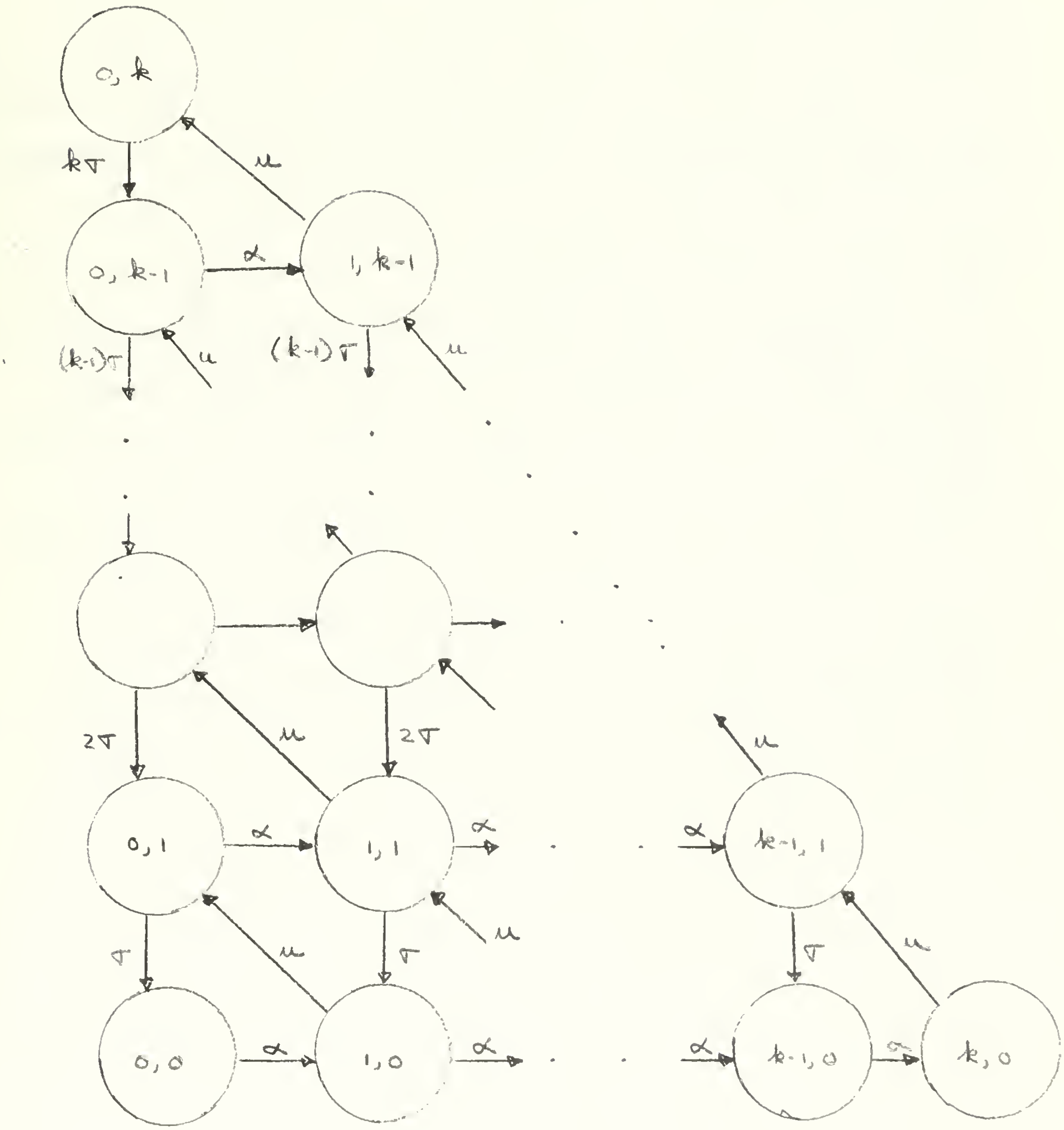


FIGURE 11

distributed, entry into the waiting line for Stage 1 (call transfered from one operator to another) is permitted only if system size (number being served in Stage 2 plus number waiting in Stage 1 plus 1 or 0 being served in Stage 1) is less than k.

3. <u>System State Diagram</u>.

Define p_{ij} to be the steady-state probability of having i calls waiting to be patched or being patched by the operator and j calls in progress. The state diagram for this system is shown in Figure 12 (note that i + j cannot be greater than k). The probability of r lines being busy (where $0 \leq r \leq k$) is the sum of the state probabilities of the diagonal where i + j = r.



0, k
kσ
u
0, k-1
α
1, k-1
(k-1)σ
u
(k-1)σ
u
2σ
u
2σ
u
0, 1
α
1, 1
α
α
k-1, 1
σ
u
σ
u
σ
u
0, 0
α
1, 0
α
α
k-1, 0
σ
k, 0

FIGURE 12

4. System Transition Matrix.

A precise, general analytical solution of this system does not appear possible. This will be more apparent later. The state diagram can be broken into $k + 1$ "super-states", each super-state being the probability of r lines, $0 \leq r \leq k$, being busy. While all states within a super-state have the same probability, $\alpha \Delta t$, of moving to the next higher super-state in Δt, each has a different rate of movement to a lower state. Transition within states of a super-state is at equal rates u. Thus the probability of going from super-state r to $r - 1$ is dependent on which particular state of the super-state is the departure point. A transition matrix for rate of movement between super-states can be obtained as shown in Figure 13. The d_r terms are unknown constants, which could possibly depend upon α, σ and u.

The number of lines occupied is a "birth and death process"[10]; movement is permitted only to adjacent states of the stochastic process at any instant of time. An expression for π_r, the steady-state probability of having r lines busy regardless of the state of the system when first observed is

$$\pi_r = \frac{\alpha_{r-1} \cdot \alpha_{r-2} \cdots \alpha_0}{d_r \sigma_r \cdot d_{r-1} \sigma_{r-1} \cdots d_1 \sigma_1} \qquad 1 \leq r \leq k \tag{1}$$

or

$$\pi_r = \frac{1}{f_r} \left(\frac{\alpha}{\sigma} \right)^r \pi_0 \tag{2}$$

where $f_r = (d_r \cdot d_{r-1} \cdots d_1)$.

$$
\begin{array}{c|c|c|c|c|c|c|c|}
 & 0 & 1 & 2 & 3 & & k-1 & k \\
\hline
0 & -\alpha & \alpha & & & & & \\
\hline
1 & d_1\sigma & -(\alpha + d_1\sigma) & \alpha & & & & \\
\hline
2 & & d_2\sigma & -(\alpha + d_2\sigma) & \alpha & & & \\
\hline
3 & & & d_3\sigma & -(d_3\sigma + \alpha) & \alpha & & \\
\hline
 & & & & & \ddots & & \\
\hline
k+1 & & & & & d_{k-1}\sigma & -(\alpha + d_{k-1}\sigma) & \alpha \\
\hline
k & & & & & & d_k\sigma & -d_k\sigma \\
\hline
\end{array}
$$

SUPER-STATE TRANSITION MATRIX

FIGURE 13

5. Determination of f_r.

The matrix above was solved by computer for 6 sets of parameters. From the results obtained for the vector $\vec{\pi}$, values for f_r were obtained by noting (2) can be written

$$f_r = \frac{\pi_0}{\pi_r} \left(\frac{\alpha}{\sigma}\right)^r \tag{3}$$

Figure 14 shows a plot of f_r against r for these solutions.

6. Error introduced by approximations.

To test the solutions obtained with equation 4-2 by using the approximations mentioned previously (i.e., assuming blocking does not seriously affect the distribution of inter departure times and disregarding the effects of priority arrivals), systems were simulated using the same parameters. Figures 15 through 17 compare the exact (simulated) probabilities of channels being used with the approximate solutions. In addition, a sensitivity analysis was performed on the approximate solutions to test the variation in blocking probability with changes in α. These results are shown in Figure 18. The implications of these results are far-reaching. It appears that the probability of blocking is a linear function of arrival rate, α. The upper boundary on α would be the minimum of patching rate or k times the calling rate. This information would be useful to the systems planner who is confronted with fluctuations in arrival rate over time.

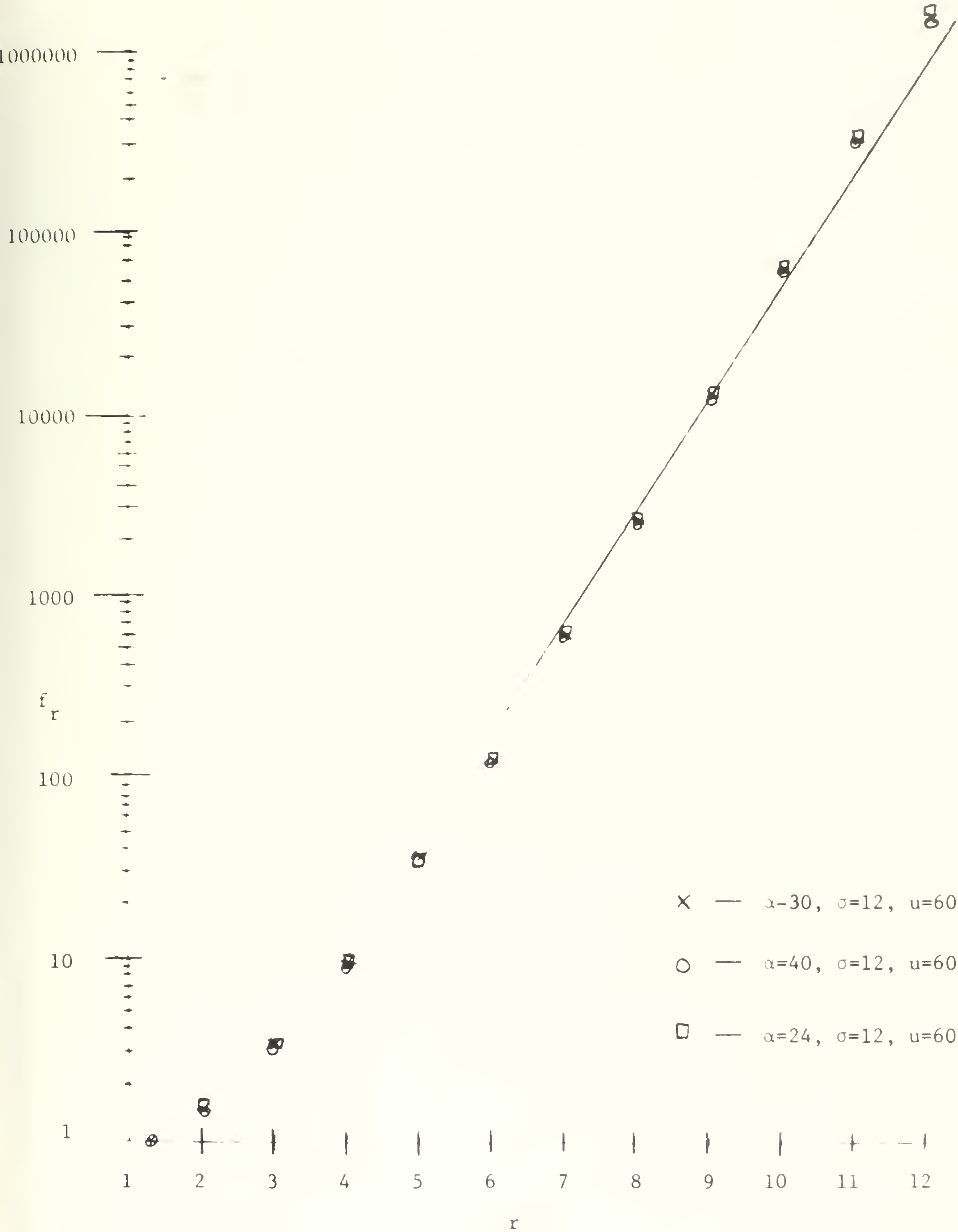
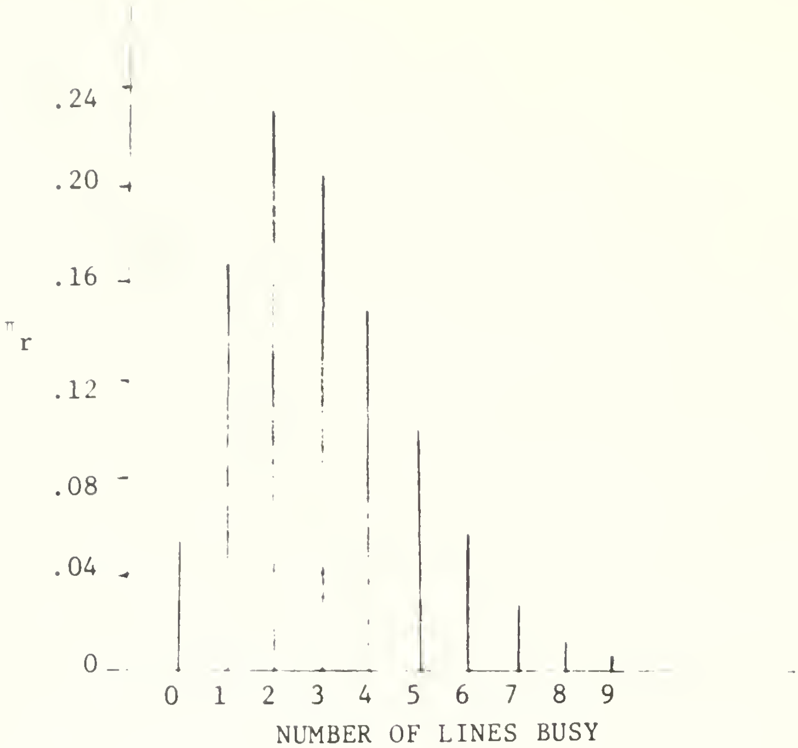
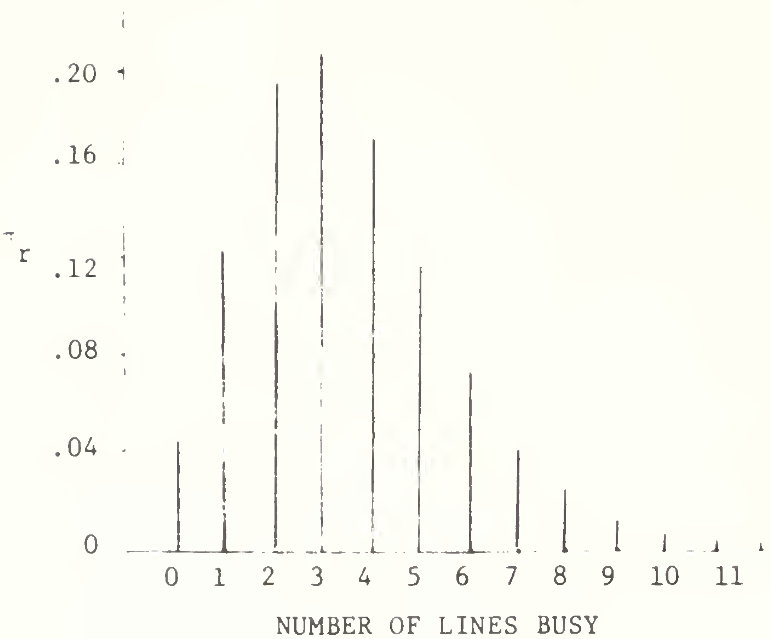

1000000
100000
10000
1000
100
10
1
f_r
1
2
3
4
5
6
7
8
9
10
11
12
r
× — α-30, σ=12, u=60
○ — α=40, σ=12, u=60
□ — α=24, σ=12, u=60

FIGURE 14



ACTUAL

α= 30
σ= 12
u= 60
k= 12



APPROXIMATE

α= 30
σ= 12
u= 60
k= 12

FIGURE 15

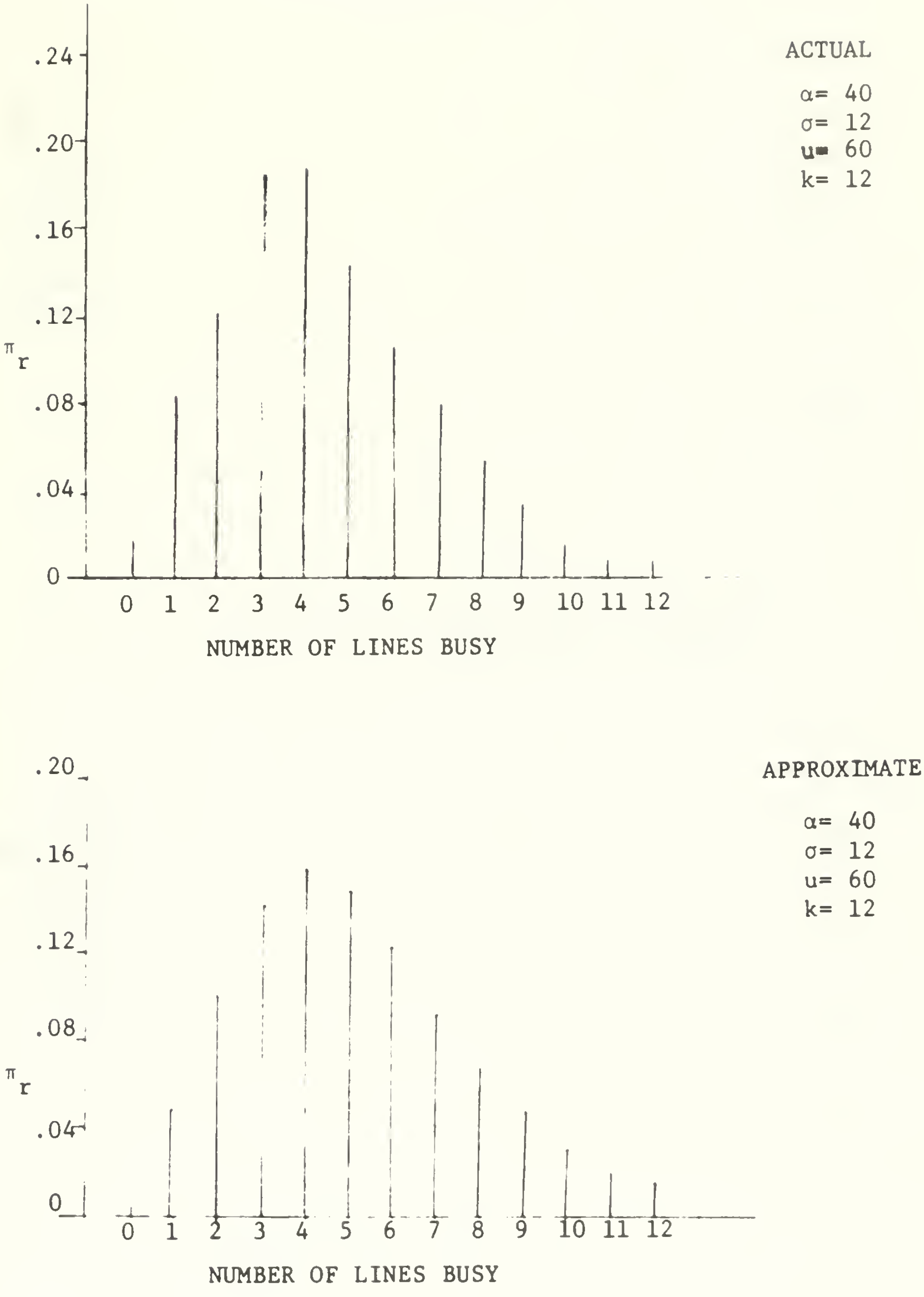
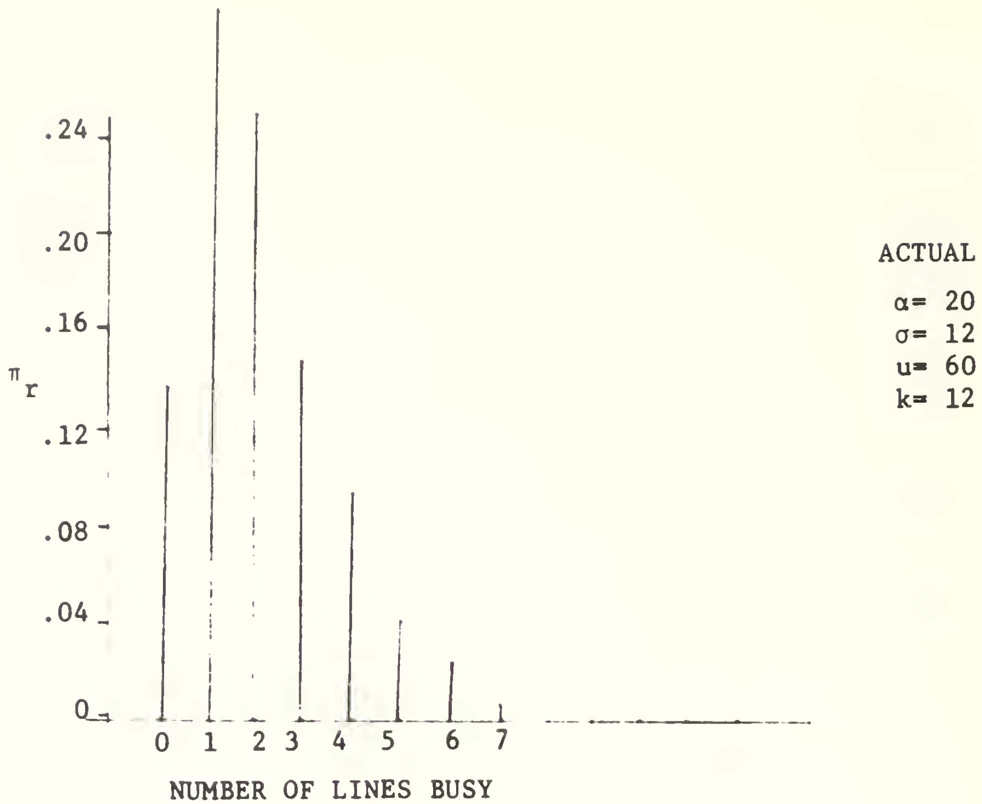

ACTUAL
α= 40
σ= 12
u= 60
k= 12
.24
.20
.16
.12
.08
.04
0
π_r
0 1 2 3 4 5 6 7 8 9 10 11 12
NUMBER OF LINES BUSY
APPROXIMATE
α= 40
σ= 12
u= 60
k= 12
.20
.16
.12
.08
.04
0
π_r
0 1 2 3 4 5 6 7 8 9 10 11 12
NUMBER OF LINES BUSY

FIGURE 16


.24
.20
.16
.12
.08
.04
0
π_r
0 1 2 3 4 5 6 7
NUMBER OF LINES BUSY
ACTUAL
α= 20
σ= 12
u= 60
k= 12

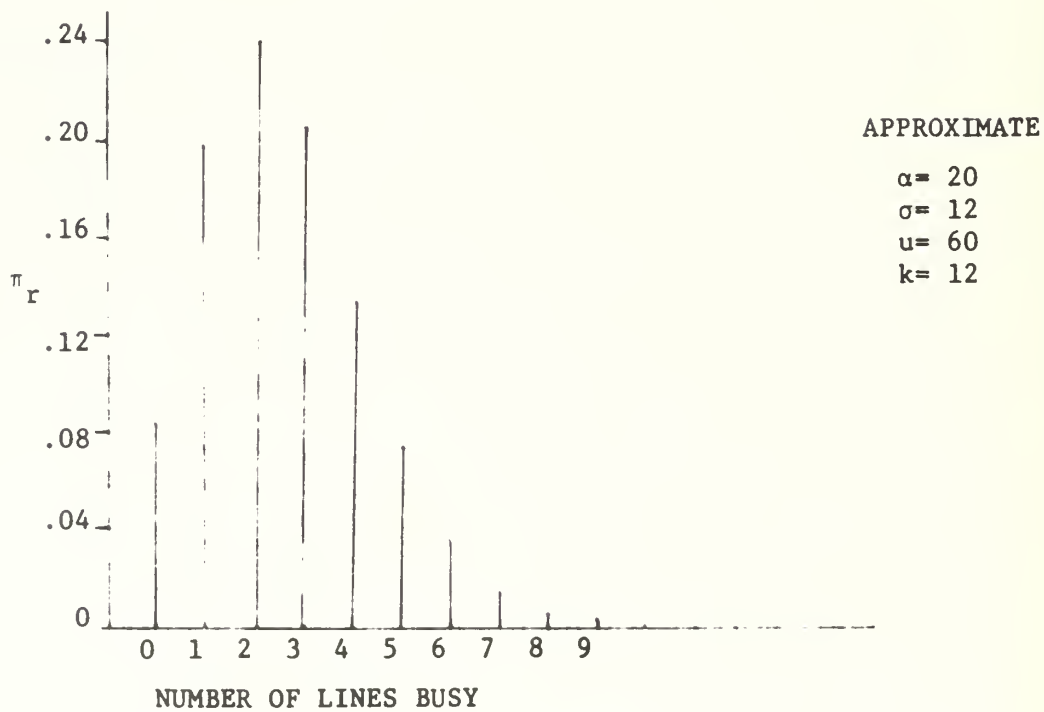

.24
.20
.16
.12
.08
.04
0
π_r
0 1 2 3 4 5 6 7 8 9
NUMBER OF LINES BUSY
APPROXIMATE
α= 20
σ= 12
u= 60
k= 12

FIGURE 17

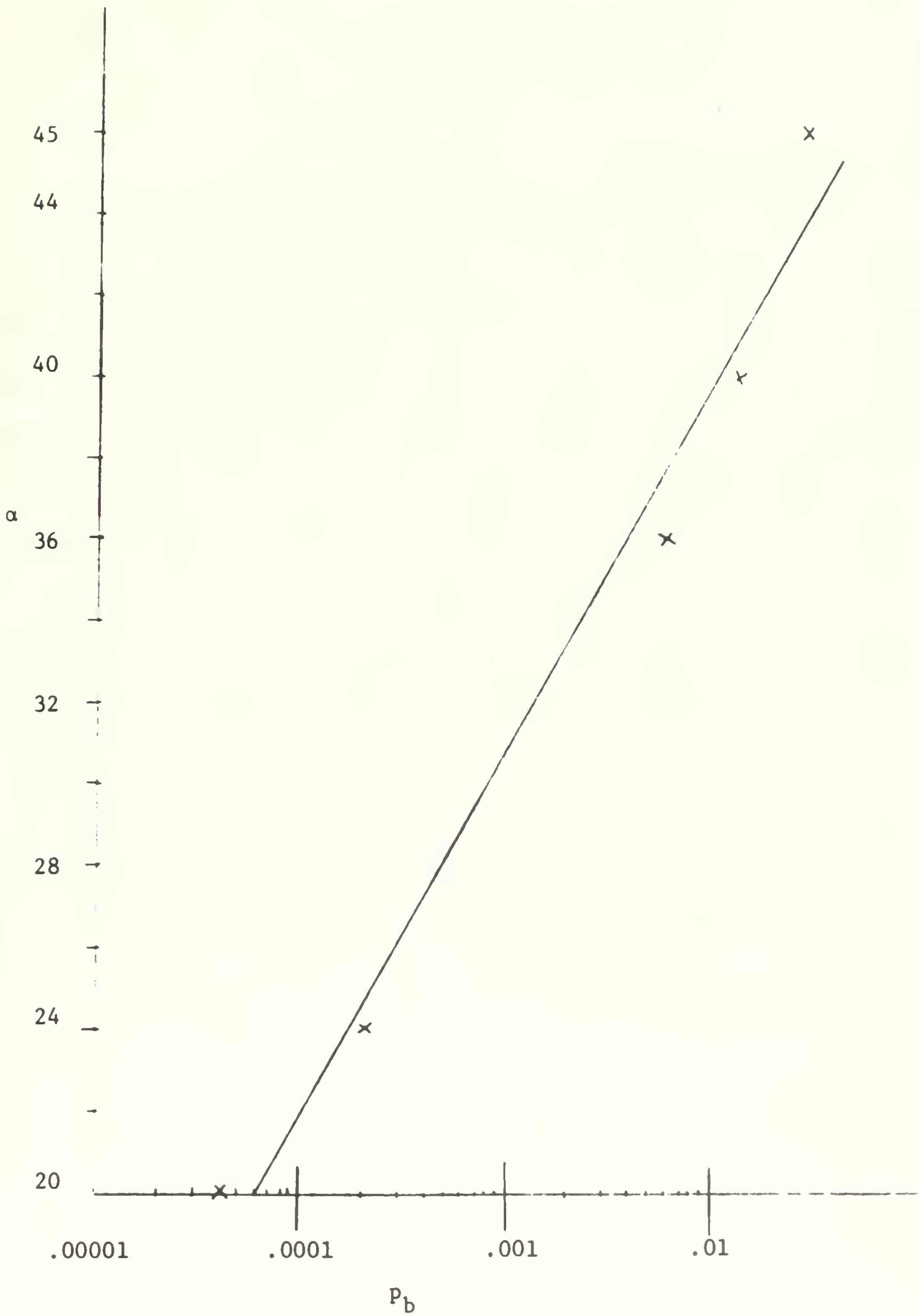


FIGURE 18

CHAPTER V

3 TERMINAL NETWORKS

1. Simulation results.

A simulation of a 3-terminal network was run to determine the effects on blocking probability of allowing routing from terminal x through terminal y to terminal z when all k lines between x and z are busy. Intuition would indicate that the mean number of channels busy for each pair of the 3-terminal network would be higher than that for a 2-terminal network with the same parameters. This is due to the inefficiency of routing (i.e., using 2 channels in tandem to route one call when the direct route is blocked) in the 3-terminal system. However, it could be expected that the probability of blocking for any call would be substantially reduced because at least 2 links must be at full capacity to block a call. This would happen despite higher individual link blocking probabilities. Table 2 lists the $\vec{\pi}$ vector for each link of the 3 terminal system and that of a 2 terminal system with the same parameters. Results from the simulation showed that 78% of calls in 3-terminal networks which were initially blocked could be indirectly routed to their destination immediately.

2. n - terminal networks with alternate routing.

It would appear that an upper limit exists for the amount of alternate routing that can be used in a general, n-terminal network. The inefficiency of the system should increase, as a function of n,

	2-TERMINAL SYSTEM	3-TERMINAL LINK 1	3-TERMINAL LINK 2	3-TERMINAL LINK 3
π_0	.0019	.0014	.0009	.0030
π_1	.0049	.0128	.0099	.0100
π_2	.0319	.0381	.0298	.0356
π_3	.0618	.0648	.0690	.0658
π_4	.0917	.1109	.1202	.1130
π_5	.1565	.1406	.1361	.1391
π_6	.1525	.1401	.1376	.1386
π_7	.1326	.1396	.1411	.1396
π_8	.1276	.1194	.1182	.1140
π_9	.0737	.0965	.0725	.0884
π_{10}	.0568	.0584	.0621	.0618
π_{11}	.0568	.0595	.0511	.0591
π_{12}	.0508	.0521	.0506	.0512

TABLE 2

until the probability of blocking

$$p_b = \pi_{k_1} \cdot \pi_{k_2} \cdot \pi_{k_3} \cdot \cdot \cdot \pi_{k_{n-1}} \qquad (1)$$

(where π_{k_1} is the probability of k channels busy in link 1)

is greater than the probability of blocking for a 2 terminal network with the same parameters. Efficiency is defined as the ability of the network to route calls over the minimum number of links.

CHAPTER VI

ANALYSIS OF RESULTS

1. General.

A study of Figures 15, 16, 17 indicates that the approximate solution for the distribution of number of lines busy in a 2-terminal system is satisfactory for many purposes. While the absolute difference between the actual and approximate results is small, the differences are significant and the approximate solutions should be used for systems design or operational planning only if simulation methods are unavailable. The approximate results serve a purpose; they provide "ball-park" answers for the systems planner when formerly no information appears to have been available.

The "real world" condition that after k lines are busy each operator can service one more subscriber was neglected in the approximate solution. This last service at the switchboard remains there until one of the k lines is free. Then it is patched to the other operator and frees the queue from which it came. This would have added another diagonal row of states to the state diagram, Figure 12. Consequently these state probabilities would be added to p_b and the approximate solution should be closer to the actual.

The function f_r (Figure 14) appears to be very stable for the range of α, σ and u for which the approximations are valid. Figure 14 which shows that f_r is linear over the range of r is a very important discovery. From this curve, the probability of r channels busy can be determined by equation 4-2 if σ and α are known. Extrapolation for larger values of k would be possible. It is interesting to note the similarity between the results of the 2-terminal, k-channel networks with and without operator patch-time considered. Equations 4-2 (with patch-time) and 2-3 differ only by constants. This is reasonable because the patch-time might be considered as just a portion of a larger unit of service time (patch-time and call holding-time). Consequently the systems differ only in mean service time.

2. Areas for future investigation.

Among some of the more fertile areas for further study would be an analytical solution for the 2 terminal, k-channel network without availing of the approximations used in this study. Introduction of priority arrivals from the subscribers into the models used here would also be a useful extension. It is doubtful that networks with more than 2 terminals would yield solutions analytically without benefit of over-simplifying assumptions. The nature of such assumptions would also provide a fruitful investigation.

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13. ABSTRACT

Models are derived for small comunications networks. Blocking probabilities are obtained for 2 terminal, k-channel systems and compared with simulation results.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Communications networks						
Connecting networks						
Blocking probabilities						
Queues						

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